On distinguishing Siegel cusp forms of degree two

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Abstract

In this work we establish several results on distinguishing Siegel cusp forms of degree two. In particular, a Hecke eigenform of level one can be determined by its second Hecke eigenvalue under a certain assumption. Moreover, we can also distinguish two Hecke eigenforms of level one by using L-functions.

Contents

1 Introduction

One of the fundamental problems in the theory of automorphic forms is *whether we can distinguish them by a set of eigenvalues*. In the elliptic modular forms case, this kind of question has been well studied. Particularly, in this case the question is equivalent to ask that how many Fourier coefficients are sufficient in order to determine an elliptic modular form. This was answered first in the context of congruences modulo a prime by Sturm [Stu87]. Later, Ghitza [Ghi11] gave a stronger result by considering two cuspidal Hecke eigenforms of distinct weights, which improved the result by Ram Murty [Mur97]. Recently, Vilardi and Xue [VX18] gave an even stronger result for two eigenforms of level one under certain assumptions.

However, distinguishing Siegel cusp forms was a long-standing unanswered problem and only recently Schmidt [Sch18], in a remarkable paper, gave an affirmative answer to this question for

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normalized eigenvalues of a Siegel cuspidal eigenform of degree two. This result has been improved by Kumar, Meher and Shankhadhar [KMS21], in which they essentially showed that any set of eigenvalues (normalized or non-normalized) at primes p of positive upper density are sufficient to determine the Siegel cuspidal eigenform. In this work we further investigate the question on distinguishing Siegel cusp forms of degree two from various aspects with several improved results.

Let $\mathcal{S}_k(\Gamma_0(N))$ be the space of Siegel cusp form of level $\Gamma_0(N)$ and weight k, where $\Gamma_0(N)$ is the Siegel congruence subgroup of level N defined as in (7). Let $F \in \mathcal{S}_k(\Gamma_0(N))$ be a Hecke eigenform with eigenvalue $\lambda_F(n)$ for $(n, N) = 1$. Then our first main result is as follows.

Theorem 1.1. *Let k_1, k_2 be distinct positive integers larger than 2. Let $F \in \mathcal{S}_{k_1}(\Gamma_0(N))$ and $G \in \mathcal{S}_{k_2}(\Gamma_0(N))$ be Hecke eigenforms. Then we can find n satisfying*

$$n \leq (2 \log N + 2)^4$$

such that $\lambda_F(n) \neq \lambda_G(n)$.

We remark that it was shown in [GS14, Corollary 5.3] that there exists some n satisfying $n \leq (2 \log N + 2)^6$ such that $\lambda_F(n) \neq \lambda_G(n)$. In particular, we obtain an improved bound for n in Theorem 1.1.

Next, we assume that $N = 1$, and let $\Gamma_2 = \mathrm{Sp}(4, \mathbb{Z})$. It is well known that the space $\mathcal{S}_k(\Gamma_2)$ has a natural decomposition into orthogonal subspaces

$$\mathcal{S}_k(\Gamma_2) = \mathcal{S}_k^{(\mathbf{P})}(\Gamma_2) \oplus \mathcal{S}_k^{(\mathbf{G})}(\Gamma_2) \tag{1}$$

with respect to the Petersson inner product. Here, $\mathcal{S}_k^{(\mathbf{P})}(\Gamma_2)$ is the subspace of Saito-Kurokawa liftings, and $\mathcal{S}_k^{(\mathbf{G})}(\Gamma_2)$ is the subspace of non-liftings. We refer the reader to [Sch18, § 2.1] for further comments related to this type decomposition. Then we can prove the following theorem:

Theorem 1.2. *Let $k_1, k_2 \in \mathcal{K}^{(\mathbf{P})}(2) \cap \mathcal{K}^{(\mathbf{G})}(2)$ be two even positive integers, where k_1 and k_2 may equal. Let $F \in \mathcal{S}_{k_1}(\Gamma_2)$ and $G \in \mathcal{S}_{k_2}(\Gamma_2)$ be Hecke eigenforms. If $\lambda_F(2) = \lambda_G(2)$, then $F = c \cdot G$ for some non-zero constant c.*

Here, the notation of $\mathcal{K}^{(*)}(2)$ can be found in (26). We remark that $\mathcal{K}^{(*)}(2)$ is a weak version of the generalized Maeda's conjecture for $\Gamma_2 = \mathrm{Sp}(4, \mathbb{Z})$. In fact, Maeda's conjecture for $\Gamma_1 = \mathrm{SL}(2, \mathbb{Z})$ would imply that $\mathcal{K}^{(\mathbf{P})}(2) = \{k \colon k \text{ even and } k \geq 10\}$. Moreover, it is expected that the set $\mathcal{K}^{(\mathbf{G})}(2)$ has the natural density of 1. See [HM97, GM12] for more discussions about Maeda's conjecture.

In addition, we can also distinguish Hecke eigenforms in each type by using L-functions with different methods. First, recall that Saito-Kurokawa liftings of level one and weight $k \in 2\mathbb{Z}_{>0}$ can be obtained from elliptic cusp forms of level one and weight $2k-2$. More precisely, let $f \in \mathcal{S}_{2k-2}(\Gamma_1)$ be a Hecke eigenform, and let π_f be the automorphic cuspidal representation of $\mathrm{GL}(2, \mathbb{A})$ associated to f. Here, $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$. Then the resulting Saito-Kurokawa lifting is in $\mathcal{S}_k(\Gamma_2)$, denoted by F_f, and F_f is also a Hecke eigenform; see [Kur78, Maa79] for more details about the classical Saito-Kurokawa liftings. The normalized spinor L-function of F_f and the normalized L-function of f are connected by the following relation:

$$L(s, \pi_{F_f}, \rho_4) = \zeta(s + 1/2)\zeta(s - 1/2)L(s, \pi_f), \tag{2}$$

where ρ_4 is the 4-dimensional irreducible representation of $\mathrm{Sp}(4, \mathbb{C})$, and π_{F_f} is the automorphic cuspidal representation of $\mathrm{GSp}(4, \mathbb{A})$ corresponding to F_f. Let ξ be a primitive Dirichlet character, and let χ be the corresponding Hecke character of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$. Let σ_1 be the standard representation of the dual group $\mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times$. Then we can define the twisted L-function by

$$L(s, \pi_{F_f} \times \chi, \rho_4 \otimes \sigma_1) = L(s + 1/2, \chi) L(s - 1/2, \chi) L(s, \pi_f \otimes \chi). \tag{3}$$

Theorem 1.3. *Let k_1, k_2 be even positive integers and $f \in \mathcal{S}_{2k_1-2}(\Gamma_1), g \in \mathcal{S}_{2k_2-2}(\Gamma_1)$ be normalized Hecke eigenforms. Suppose that there exists a non-zero constant c such that*

$$L(1/2, \pi_{F_f} \times \chi_d, \rho_4 \otimes \sigma_1) = c \cdot L(1/2, \pi_{F_g} \times \chi_d, \rho_4 \otimes \sigma_1)$$

for almost all quadratic Hecke characters χ_d of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$, which are corresponding to primitive quadratic Dirichlet characters ξ_d of conductor d. Then $k_1 = k_2$ and $F_f = F_g$.

To prove this theorem, it suffices to show that $f = g$, which is due to [LR97, Theorem B].

Finally, we will distinguish Hecke eigenforms of level one which are non-liftings. Let k_1, k_2 be even integers. Let $F \in \mathcal{S}_{k_1}^{(\mathbf{G})}(\Gamma_2)$ and $G \in \mathcal{S}_{k_2}^{(\mathbf{G})}(\Gamma_2)$ be Hecke eigenforms, and let π_F (resp. π_G) be the automorphic cuspidal representation of $\mathrm{GSp}(4, \mathbb{A})$ corresponding to F (resp. G). Then we can define the Rankin-Selberg L-function of F and G, denoted by $L(s, \pi_F \times \pi_G, \rho_i \otimes \rho_j)$ with $i, j \in \{4, 5\}$; see [PSS14, (271)]. (Note that the L-function here is actually the finite part of L-functions in [PSS14].) Moreover, $L(s, \pi_F \times \pi_G, \rho_i \otimes \rho_j)$ has a pole at $s = 1$ if and only if $i = j, k_1 = k_2$ and $F = c \cdot G$ for some non-zero constant c; see [PSS14, Theorem 5.2.3].

Theorem 1.4. *Assume the notations above. Suppose that $L(s, \pi_F \times \pi_G, \rho_4 \otimes \rho_4)$ and $L(s, \pi_F \times \pi_F, \rho_4 \otimes \rho_4)$ satisfy the Generalized Riemann hypothesis. If F is not a scalar multiplication of G, then there exists an integer*

$$n \ll (\log k_1 k_2)^2 (\log \log k_1 k_2)^4$$

such that $\tilde{\lambda}_F(n) \neq \tilde{\lambda}_G(n)$. Here, $\tilde{\lambda}_F(n) = n^{3/2-k_1} \lambda_F(n)$ (resp. $\tilde{\lambda}_G(n) = n^{3/2-k_2} \lambda_G(n)$) is the normalized Hecke eigenvalue for F (resp. G).

To prove above theorem, we will apply the method in [GH93]. Then combine with Lemma 6.1 and we will conclude Theorem 1.4.

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2 Preliminaries

We let

$$\mathrm{GSp}(4) \coloneqq \{g \in \mathrm{GL}(4) \colon {}^t g J g = \mu(g) J,\ \mu(g) \in \mathrm{GL}(1)\}, \quad J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}. \tag{4}$$

The function μ is called the multiplier homomorphism. The kernel of this function is the symplectic group $\mathrm{Sp}(4)$. Let Z be the center of $\mathrm{GSp}(4)$ and $\mathrm{PGSp}(4) = \mathrm{GSp}(4)/\mathrm{Z}$. When speaking about Siegel modular forms of degree 2, it is more convenient to realize symplectic groups using the symplectic form $J = \left[\begin{smallmatrix} 0 & 1_2 \\ -1_2 & 0 \end{smallmatrix}\right]$. The Siegel upper half plane of degree 2 is defined by

$$\mathbb{H}_2 \coloneqq \{Z \in \mathrm{Mat}_2(\mathbb{C})\colon\ {}^tZ = Z, \mathrm{Im}(Z) > 0\}. \tag{5}$$

The group $\mathrm{GSp}(4,\mathbb{R})^+ \coloneqq \{g \in \mathrm{GSp}(4,\mathbb{R})\colon \mu(g) > 0\}$ acts on $\mathbb{H}_2$ by

$$g\langle Z\rangle \coloneqq (AZ+B)(CZ+D)^{-1} \quad \text{for } g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{GSp}(4,\mathbb{R})^+ \text{ and } Z \in \mathbb{H}_2. \tag{6}$$

Let $\Gamma_2 = \mathrm{Sp}(4,\mathbb{Z})$. In general, for a positive integer N we let

$$\Gamma_0(N) \coloneqq \left\{\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{Sp}(4,\mathbb{Z})\colon C \equiv 0 \pmod N\right\} \tag{7}$$

be the Siegel congruence subgroup of level N. It is clear that $\Gamma_2 = \Gamma_0(1)$.

Let $\mathcal{M}_k(\Gamma_0(N))$ be the space of Siegel modular form of weight k with respect to $\Gamma_0(N)$, and let $\mathcal{S}_k(\Gamma_0(N))$ be the subspace of cusp forms. That is to say, for any function $F \in \mathcal{M}_k(\Gamma_0(N))$, it is a holomorphic $\mathbb{C}$-valued function on $\mathbb{H}_2$ satisfying $\big(F|_k\gamma\big)(Z) = F(Z)$ for all $\gamma \in \Gamma_0(N)$. Here,

$$\big(F|_kg\big)(Z) \coloneqq \mu(g)^k j(g,Z)^{-k} F(g\langle Z\rangle) \quad \text{for } g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{GSp}(4,\mathbb{R})^+ \text{ and } Z \in \mathbb{H}_2, \tag{8}$$

where $j(g,Z) \coloneqq \det(CZ+D)$ is the automorphy factor. We remark that this operator differs from the classical one used in [And74] by a factor. We do so to make the center of $\mathrm{GSp}(4,\mathbb{R})^+$ act trivially.

Let $F \in \mathcal{S}_k(\Gamma_0(N))$ be a Hecke eigenform, i.e., it is an eigenvector for all the Hecke operator $T(n), (n,N) = 1$. Denote by $\lambda_F(n)$ the eigenvalue of F under $T(n)$ when $(n,N) = 1$. For any prime $p \nmid N$, we let $\alpha_{p,0}, \alpha_{p,1}, \alpha_{p,2}$ be the classical Satake parameters of F at p. It is well known that

$$\alpha_{p,0}^2\alpha_{p,1}\alpha_{p,2} = p^{2k-3}. \tag{9}$$

In particular, let $N = 1$ and $F \in \mathcal{S}_k(\Gamma_2)$ be a Hecke eigenform, we can define the L-series

$$H(s) = \sum_{n=1}^{\infty} \frac{\lambda_F(n)}{n^s}. \tag{10}$$

This can be written as a Euler product

$$H(s) = \prod_p H_p(s) = \prod_p \left(1 + \frac{\lambda_F(p)}{p^s} + \frac{\lambda_F(p^2)}{p^{2s}} + \cdots\right) \tag{11}$$

provided $\Re(s) > k$. Moreover, one can show that

$$H_p(s) = \big(1 - p^{2k-4-2s}\big) L_p(s,F,\mathrm{spin}), \tag{12}$$

where $L_p(s,F,\mathrm{spin})$ is the local spinor L-factor of F at p and it can be given:

$$L_p(s,F,\mathrm{spin})^{-1} = (1-\alpha_{p,0}p^{-s})(1-\alpha_{p,0}\alpha_{p,1}p^{-s})(1-\alpha_{p,0}\alpha_{p,2}p^{-s})(1-\alpha_{p,0}\alpha_{p,1}\alpha_{p,2}p^{-s}). \tag{13}$$

On the other hand, by [And74, p. 62, 69] one can see that

$$L_p(s,F,\mathrm{spin})^{-1} = 1 - \lambda_F(p)p^{-s} + (\lambda_F(p)^2 - \lambda_F(p^2) - p^{2k-4})p^{-2s} - \lambda_F(p)p^{2k-3-3s} + p^{4k-6-4s}. \tag{14}$$

In this case, we can define the spinor L-function

$$L(s,F,\mathrm{spin})=\prod_p L_p(s,F,\mathrm{spin}).$$

The analytic property of the spinor L-function can be found in [And74]. Let $\alpha_p=p^{3/2-k}\alpha_{p,0}$ and $\beta_p=\alpha_p\alpha_{p,1}$. By comparing (13) with (14), we obtain (also see [PS09, Proposition 4.1])

$$\lambda_F(p)=p^{k-3/2}(\alpha_p+\alpha_p^{-1}+\beta_p+\beta_p^{-1}), \tag{15}$$
$$\lambda_F(p^2)=p^{2k-3}\left((\alpha_p+\alpha_p^{-1})^2+(\alpha_p+\alpha_p^{-1})(\beta_p+\beta_p^{-1})+(\beta_p+\beta_p^{-1})^2-2-1/p\right). \tag{16}$$

Let ρ_4 be the 4-dimensional irreducible representation of $\mathrm{Sp}(4,\mathbb{C})$. In fact, ρ_4 is the natural representation of $\mathrm{Sp}(4,\mathbb{C})$ on $\mathbb{C}^4$, which is also called the spin representation. For later use, we would normalize the spinor L-function. More precisely, the normalized spinor L-function $L(s,\pi_F,\rho_4)$ is defined as follows:

$$L(s,\pi_F,\rho_4)=L(s+k-3/2,F,\mathrm{spin})=\sum_{n=1}^{\infty}\frac{a_F(n)}{n^s}. \tag{17}$$

Actually, this is the finite part of the completed L-function of π_F, where π_F is the automorphic cuspidal representation of $\mathrm{GSp}(4,\mathbb{A})$ associated to F. For more details about the connection between Siegel modular forms of degree 2 and automorphic representations of $\mathrm{GSp}(4,\mathbb{A})$; see [AS01] and [Sch17, Section 4.2]. Moreover, let $\tilde{\lambda}_F(n)=n^{3/2-k}\lambda_F(n)$ be the normalized eigenvalues. It follows that

$$\sum_{n=1}^{\infty}\frac{\tilde{\lambda}_F(n)}{n^s}=\zeta(2s+1)^{-1}L(s,\pi_F,\rho_4). \tag{18}$$

Note that if $F\in\mathcal{S}_k(\Gamma_0(N))$ with $N>1$, we still can define the partial spinor L-functions by Euler products for all primes p not dividing N. In particular, the local factor at p with $(p,N)=1$ is defined in the same way as above.

Similarly, let ρ_5 be the 5-dimensional irreducible representation of $\mathrm{Sp}(4,\mathbb{C})$. An explicit formula for the representation ρ_5 as a map $\mathrm{Sp}(4,\mathbb{C})\to\mathrm{SO}(5,\mathbb{C})$ is given in [RS07, Appendix A.7]. The standard L-function associated to F is defined as

$$L(s,\pi_F,\rho_5)=\prod_p L_p(s,F,\mathrm{std})=\sum_{n=1}^{\infty}\frac{b_F(n)}{n^s}, \tag{19}$$

where

$$L_p(s,F,\mathrm{std})^{-1}=(1-p^{-s})(1-\alpha_{p,1}p^{-s})(1-\alpha_{p,2}p^{-s})(1-\alpha_{p,1}^{-1}p^{-s})(1-\alpha_{p,2}^{-1}p^{-s}). \tag{20}$$

One can also refer to [Böc85, PS09] for the analytic properties of the standard L-functions.

3 Proof of Theorem 1.1

First, by (12) and (14) we have

$$\lambda_F(p^3)=2\lambda_F(p)\lambda_F(p^2)-\lambda_F(p)^3+\lambda_F(p)(p^{2k-3}+p^{2k-4}), \tag{21}$$

and

$$\lambda_F(p^4) = -\lambda_F(p)^4 + \lambda_F(p)^2\lambda_F(p^2) + \lambda_F(p^2)^2 + \lambda_F(p)^2p^{2k-4} + \lambda_F(p^2)p^{2k-4} + 2\lambda_F(p)^2p^{2k-3} - p^{4k-6}. \tag{22}$$

Then we can prove the following result:

Theorem 3.1. *Let k_1, k_2 be distinct positive integers larger than 2. Let $F \in \mathcal{S}_{k_1}(\Gamma_0(N))$ and $G \in \mathcal{S}_{k_2}(\Gamma_0(N))$ be Hecke eigenforms. Then for any prime p not dividing N, we can find $i \in \{1, 2, 3, 4\}$ such that*

$$\lambda_F(p^i) \neq \lambda_G(p^i).$$

To prove this theorem, we need the following lemma.

Lemma 3.2. *Let $F \in \mathcal{S}_k(\Gamma_0(N))$ be a Hecke eigenform. Let $p \nmid N$ be a prime. If $\lambda_F(p) = 0$, then*

$$|\lambda_F(p^2)| < p^{2k-2} + 2p^{2k-4}.$$

Proof. By (15)-(16) and $\lambda_F(p) = 0$, we have

$$\begin{aligned}\lambda_F(p^2) &= \lambda_F(p)^2 - p^{2k-3}((\alpha_p + \alpha_p^{-1})(\beta_p + \beta_p^{-1}) + 2 + 1/p) \\ &= p^{2k-3}((\alpha_p + \alpha_p^{-1})(\alpha_p + \alpha_p^{-1}) - 2 - 1/p) \\ &= p^{2k-3}(\alpha_p^2 + \alpha_p^{-2} - 1/p).\end{aligned}$$

Observe that α_p is just the $\sigma(p)$ as in [PS09, Theorem 3.2]. In particular, we have $1 \leq |\alpha_p| < p^{1/2}$. Hence, we obtain the desired estimate. □

Proof of Theorem 3.1. Assume that there exists a prime $p \nmid N$ such that $\lambda_F(p^i) = \lambda_G(p^i)$ for $i = 1, 2, 3, 4$; we will obtain a contradiction. More precisely, we consider the following two cases:

(1) If $\lambda_F(p) \neq 0$, then by (21) and $\lambda_F(p^i) = \lambda_G(p^i)$ $(i = 1, 2, 3)$, we have

$$\lambda_F(p)(p^{2k_1-3} + p^{2k_1-4}) = \lambda_G(p)(p^{2k_2-3} + p^{2k_2-4}).$$

This yields the contradiction $k_1 = k_2$.

(2) If $\lambda_F(p) = 0$, then $\lambda_G(p) = 0$ by assumption. By Lemma 3.2 we have

$$|\lambda_F(p^2)| < p^{2k_1-2} + 2p^{2k_1-4} \quad \text{and} \quad |\lambda_G(p^2)| < p^{2k_2-2} + 2p^{2k_2-4}.$$

Without loss of generality, we assume that $k_1 \geq k_2 + 1$. Since $\lambda_F(p^2) = \lambda_G(p^2)$, we have

$$|\lambda_F(p^2)| = |\lambda_G(p^2)| < 2p^{2k_2-2} + p^{2k_2-4}. \tag{23}$$

On the other hand, it follows from (22) and $\lambda_F(p^i) = \lambda_G(p^i), i = 1, 2, 3, 4$, that

$$\lambda_F(p^2)p^{2k_1-4} - p^{4k_1-6} = \lambda_G(p^2)p^{2k_2-4} - p^{4k_2-6} \tag{24}$$

Then we have $\lambda_F(p^2)(p^{2k_1-4} - p^{2k_2-4}) = p^{4k_1-6} - p^{4k_2-6}$. Multiplying p^2 both sides we obtain

$$\lambda_F(p^2)(p^{2k_1-2} - p^{2k_2-2}) = p^{4k_1-4} - p^{4k_2-4} = (p^{2k_1-2} - p^{2k_2-2})(p^{2k_1-2} + p^{2k_2-2}). \tag{25}$$

It follows that $\lambda_F(p^2) = p^{2k_1-2} + p^{2k_2-2}$. This equality leads to a contradiction due to (23) and $k_1 \geq k_2 + 1$. □

Then Theorem 1.1 immediately follows from Theorem 3.1 and Lemma 3.3 below.

Lemma 3.3 ([Ghi11, cf. § 2])**.** *Let $N \geq 1$ be a positive integer, then we can find a prime p such that $(p, N) = 1$ and $p \leq 2 \log N + 2$.*

4 Proof of Theorem 1.2

In this section, we only consider $\Gamma_2 = \mathrm{Sp}(4,\mathbb{Z})$. Let $m_k = \dim_{\mathbb{C}} \mathcal{S}_k(\Gamma_2)$, and let

$$\mathcal{K}^{(*)}(2) = \{k \in \mathbb{Z}\colon \text{The characteristic polynomial of } T_k(2) \text{ for } F \in \mathcal{S}_k^{(*)}(\Gamma_2) \text{ is irreducible}\}, \quad (26)$$

where $\mathcal{S}_k^{(*)}(\Gamma_2)$ is the set of those $F \in \mathcal{S}_k(\Gamma_2)$ of type $(*)$ as in (1). Let $m_k^{(*)} = \dim_{\mathbb{C}} \mathcal{S}_k^{(*)}(\Gamma_2)$. Evidently, $m_k = m_k^{(\mathbf{P})} + m_k^{(\mathbf{G})}$. It is well known that $m_k^{(\mathbf{P})} > 0$ only if $k \in \mathbb{Z}_{\geq 10}$ is even and $m_k^{(\mathbf{G})} > 0$ only if $k \in \mathbb{Z}_{\geq 20}$. We also note that $m_k^{(\mathbf{P})} = \dim_{\mathbb{C}} \mathcal{S}_{2k-2}(\Gamma_1)$, where $\Gamma_1 = \mathrm{SL}(2,\mathbb{Z})$.

Proof of Theorem 1.2. We are going to separate into the following three cases.

(1) If F and G both are Saito-Kurokawa liftings, say $F \in \mathcal{S}_{k_1}^{(\mathbf{P})}(\Gamma_2)$ and $G \in \mathcal{S}_{k_2}^{(\mathbf{P})}(\Gamma_2)$ with $k_1, k_2 \in \mathcal{K}^{(\mathbf{P})}(2)$, then we can write $F = F_f$ and $G = F_g$, which are lifts from $f \in \mathcal{S}_{2k_1-2}(\Gamma_1)$ and $g \in \mathcal{S}_{2k_2-2}(\Gamma_1)$, respectively. Recall that if f and g are Hecke eigenforms, then both F_f and F_g are also Hecke eigenforms. Let $T_{2k-2}^{(1)}(2)$ be the Hecke operator on $f \in \mathcal{S}_{2k-2}(\Gamma_1)$ with Hecke eigenvalue $\lambda_f(2)$, and let $T_k(2)$ be the Hecke operator on $F_f \in \mathcal{S}_k(\Gamma_2)$ with Hecke eigenvalue $\lambda_{F_f}(2)$. Then we have

$$\lambda_{F_f}(2) = 2^{k-1} + 2^{k-2} + \lambda_f(2). \quad (27)$$

Moreover, let $P(T_k^{(\mathbf{P})}(2), t)$ be the characteristic polynomial of $T_k(2)$ on $\mathcal{S}_k^{(\mathbf{P})}(\Gamma_2)$, which is irreducible if $k \in \mathcal{K}^{(\mathbf{P})}(2)$. For $k \in \mathcal{K}^{(\mathbf{P})}(2)$ we can see that the characteristic polynomial $P(T_{2k-2}^{(1)}(2), t)$ of $T_{2k-2}^{(1)}(2)$ is irreducible as well since $P(T_k^{(\mathbf{P})}(2), t) = P(T_{2k-2}^{(1)}(2), t - 2^{k-1} - 2^{k-2})$. We can further assume that f is not a constant multiple of g since the Saito-Kurokawa lifting is injective.

(i) If $k_1 = k_2 = k$, then $\lambda_{F_f}(2) \neq \lambda_{F_g}(2)$ due to the fact that the irreducible characteristic polynomial $P(T_k^{(\mathbf{P})}(2), t)$ has distinct roots.

(ii) If $m_{k_1}^{(\mathbf{P})} \neq m_{k_2}^{(\mathbf{P})}$, then $\deg P(T_{k_1}^{(\mathbf{P})}(2), t) \neq \deg P(T_{k_2}^{(\mathbf{P})}(2), t)$. Recall that both of them are irreducible, it follows that $P(T_{k_1}^{(\mathbf{P})}(2), t)$ and $P(T_{k_2}^{(\mathbf{P})}(2), t)$ have distinct roots. Hence, $\lambda_{F_f}(2) \neq \lambda_{F_g}(2)$.

(iii) If $m_{k_1}^{(\mathbf{P})} = m_{k_2}^{(\mathbf{P})} \geq 1$ and $k_1 \neq k_2$, it is clear that $2k_1 - 2, 2k_2 - 2 \geq 18$. Additionally, we can show that there exists $n \geq 1$ such that $2k_1 - 2, 2k_2 - 2 \in \{12n+6, 12n+10, 12n+14\}$ since k_1, k_2 are even and $m_{k_1}^{(\mathbf{P})} = m_{k_2}^{(\mathbf{P})}$. On the other hand, we can show that

$$\operatorname{Tr} T_{k_1}^{(\mathbf{P})}(2) = m_{k_1}^{(\mathbf{P})}(2^{k_1-1} + 2^{k_1-2}) + \operatorname{Tr} T_{2k_1-2}^{(1)}(2). \quad (28)$$

Assume that $k_1 = k_2 + l$ with $l > 0$. Then by the choice of k_1, k_2, we know that $l \in \{2, 4\}$. Let $l = 2^m$ as in [VX18, Corollary 3.4], and so $m \in \{1, 2\}$. It follows from $m_{k_1}^{(\mathbf{P})} = m_{k_2}^{(\mathbf{P})}$ and (28) that

$$\operatorname{Tr} T_{k_1}^{(\mathbf{P})}(2) - \operatorname{Tr} T_{k_2}^{(\mathbf{P})}(2) = 2^{k_2-2}\left(m_{k_2}^{(\mathbf{P})}(2^{k_1-k_2+1} + 2^{k_1-k_2} - 3) + a_{k_2-1,l} - 2^{m+5-k_2} c_{k_2-1,l}\right),$$

where $a_{k_2-1,l}$ is an integer and $c_{k_2-1,l}$ is an odd integer as in the proof of [VX18, Corollary 3.4]. However, we know that $m + 5 - k_2 \leq -3$ since $m \leq 2$ and $k_2 \geq 10$. Therefore,

$2^{m+5-k_2}c_{k_2-1,l}$ is not an integer and hence $\operatorname{Tr} T_{k_1}^{(\mathbf{P})}(2) \neq \operatorname{Tr} T_{k_2}^{(\mathbf{P})}(2)$. By irreducibility of characteristic polynomials $\operatorname{Tr} T_{k_1}^{(\mathbf{P})}(2) = \operatorname{Tr} \lambda_{F_f}(2)$ and $\operatorname{Tr} T_{k_2}^{(\mathbf{P})}(2) = \operatorname{Tr} \lambda_{F_g}(2)$, which implies that $\lambda_{F_f}(2) \neq \lambda_{F_g}(2)$.

(2) If F and G both are non-liftings, say $F \in \mathcal{S}_{k_1}^{(\mathbf{G})}(\Gamma_2)$ and $G \in \mathcal{S}_{k_2}^{(\mathbf{G})}(\Gamma_2)$, we can apply the similar arguments as in the above case (1). More precisely, if $k_1 = k_2 \in \mathcal{K}^{(\mathbf{G})}(2)$, as the characteristic polynomial of $T_{k_1}^{(\mathbf{G})}(2)$ is irreducible by assumption, then all of its roots are distinct. Thus if $F \neq c\cdot G$ for any non-zero constant c, then $\lambda_F(2) \neq \lambda_G(2)$. On the other hand, if $k_1 \neq k_2$, then it follows from straightforward computations by using [RSY21, Theorem 3.1] that $m_{k_1}^{(\mathbf{G})} > m_{k_2}^{(\mathbf{G})}$ for any $k_1 > k_2$ and $k_1, k_2 \geq 40$. Finally, if $k_1 \neq k_2$ and $m_{k_1}^{(\mathbf{G})} = m_{k_2}^{(\mathbf{G})}$, then for we can just use [BCFvdG17] to see that $\operatorname{Tr} T_{k_1}^{(\mathbf{G})}(2) \neq \operatorname{Tr} T_{k_2}^{(\mathbf{G})}(2)$ for all small even weights $k_1, k_2 \leq 38$. Hence the assertion follows.

(3) If one of F and G is a Saito-Kurokawa lifting and the other one is non-lifting, say $F \in \mathcal{S}_{k_1}^{(\mathbf{P})}(\Gamma_2)$ and $G \in \mathcal{S}_{k_2}^{(\mathbf{G})}(\Gamma_2)$. It follows from (15) and (27) that if $k_1 - k_2 \geq 6$, then we must have $\lambda_F(2) > \lambda_G(2)$. Next, we only need to consider $k_1 - k_2 \leq 4$ cases. Again, by [RSY21, Theorem 3.1] we can easily to see that $m_{k_1}^{(\mathbf{P})} \neq m_{k_2}^{(\mathbf{G})}$ unless $k_2 \in S := \{20, 22, 24, 26, 28, 30, 32\}$. By [Bre99], we know that the Hecke eigenvalues $\lambda_F(n) > 0$ for all n. Then by irreducibility of characteristic polynomials $\operatorname{Tr} T_{k_1}^{(\mathbf{P})}(2) = \operatorname{Tr} \lambda_F(2) > 0$. On the other hand, for every $k_2 \in S$, by [BCFvdG17] we can see that $\operatorname{Tr} T_{k_2}^{(\mathbf{G})}(2) < 0$ and so $\operatorname{Tr} \lambda_G(2) = \operatorname{Tr} T_{k_2}^{(\mathbf{G})}(2) \neq \operatorname{Tr} T_{k_1}^{(\mathbf{P})}(2)$. In particular, we have $\lambda_F(2) \neq \lambda_G(2)$. Hence the assertion follows. $\square$

Since Saito-Kurokawa liftings only happen for even weights, there is no need to discuss the odd weights situation for cases (1) and (3) in the proof above. However, we still can consider the case (2), i.e., both of F and G are non-liftings with k_1 and k_2 being odd integers. In particular, with a similar argument, we can show the following result.

Corollary 4.1. *Let $k_1, k_2 \in \mathcal{K}^{(\mathbf{G})}(2)$ be two odd positive integers, where k_1 and k_2 may equal. Let $F \in \mathcal{S}_{k_1}(\Gamma_2)$ and $G \in \mathcal{S}_{k_2}(\Gamma_2)$ be Hecke eigenforms. If $\lambda_F(2) = \lambda_G(2)$, then $F = c \cdot G$ for some non-zero constant c.*

We note that our approach cannot apply for the case that k_1 and k_2 have the different parity. It would be interesting to work out a general result of Theorem 1.2 without any restriction of weights.

5 Proof of Theorem 1.3

Proof of Theorem 1.3. The proof of the theorem is based on [LR97, Theorem B]. It suffices to assume that $d < 0$. For a Saito-Kurokawa lifting F_f, by (3) we have

$$L(1/2, \pi_{F_f} \times \chi_d, \rho_4 \otimes \sigma_1) = L(0, \chi_d) L(1, \chi_d) L(1/2, \pi_f \otimes \chi_d),$$

where σ_1 is the standard representation of the dual group $\mathbb{C}^\times$. By the well known result of Dirichlet, we have $L(1, \chi_d) \neq 0$. Then by the functional equation of $L(s, \chi_d)$, we can see that $L(0, \chi_d) \neq 0$. (We have $d < 0$ and hence $\xi_d(-1) = -1$.) This gives

$$L(1/2, \pi_f \otimes \chi_d) = c \cdot L(1/2, \pi_g \otimes \chi_d)$$

for almost all quadratic Hecke characters χ_d of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$, which are corresponding to primitive Dirichlet quadratic characters ξ_d of conductor $d<0$. Recall that f is of weight $2k_1-2$ and g is of weight $2k_2-2$ with k_1,k_2 even, then the root numbers of the automorphic cuspidal representation associated to f and g are -1. Similar to [LR97, Theorem B], we also have the set

$$\mathcal{D}^\omega=\{d\colon \omega d>0, \text{ and } d\equiv v^2 \pmod{4M} \text{ for some } v \text{ coprime to } 4M \text{ and } M \text{ is an integer}\},$$

where ω is the root number. This is exactly our case since we assume that $d<0$ and $\omega=-1$. In this case, for any $d\in\mathcal{D}^\omega$, we can find a non-zero constant c such that

$$L(1/2,\pi_f\otimes\chi_d)=c\cdot L(1/2,\pi_g\otimes\chi_d),$$

By the virtual of [LR97, Theorem B], we have $k_1=k_2$ and $f=g$. Therefore, $F_f=F_g$. $\square$

6 Proof of Theorem 1.4

Proof of Theorem 1.4. We would consider the following integral

$$\frac{1}{2\pi i}\int_{(2)}\left(\frac{x^{s-\frac{1}{2}}-x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\right)^2\left(-\frac{Z'}{Z}(s)\right)ds,$$

where later we will choose $Z(s)$ to be $L(s,\pi_F\times\pi_F,\rho_4\otimes\rho_4)$ and $L(s,\pi_F\times\pi_G,\rho_4\otimes\rho_4)$. Assume that

$$-\frac{L'}{L}(s,\pi_F\times\pi_F,\rho_4\otimes\rho_4)=\sum_{n=1}^{\infty}\frac{\Lambda_{F\times F}(n)}{n^s}\quad\text{and}\quad-\frac{L'}{L}(s,\pi_F\times\pi_G,\rho_4\otimes\rho_4)=\sum_{n=1}^{\infty}\frac{\Lambda_{F\times G}(n)}{n^s}.$$

Following the idea of [GH93], for $x>0$ we can show that

$$2\sum_{n<x^2}\frac{\Lambda_{F\times F}(n)}{n^{\frac{1}{2}}}\log\left(\frac{x^2}{n}\right)=8(x-2+x^{-1})-4\sum_{\gamma}\frac{\sin^2(\gamma\log x)}{\gamma^2}+J_1, \tag{29}$$

where $\frac{1}{2}+i\gamma$ runs over the non-trivial zeros of $L(s,\pi_F\times\pi_F,\rho_4\otimes\rho_4)$ and

$$J_1=\frac{1}{2\pi i}\int_{(1/2)}\left(\frac{G_1'}{G_1}(s)+\frac{G_1'}{G_1}(1-s)\right)\left(\frac{x^{s-\frac{1}{2}}-x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\right)^2 ds.$$

Here, $G_1(s)$ is the archimedean part of $L(s,\pi_F\times\pi_F,\rho_4\otimes\rho_4)$; see Proposition A.3 with $k_1=k_2$. Note that we moved the integration line to $\mathrm{Re}(s)=1/2$ since there exist no poles of $G_1(s)$ when $1/4\le\mathrm{Re}(s)\le 3/4$ by Proposition A.3 in Appendix A.

Similarly, in the case of $L(s,\pi_F\times\pi_G,\rho_4\otimes\rho_4)$, we have

$$2\sum_{n<x^2}\frac{\Lambda_{F\times G}(n)}{n^{\frac{1}{2}}}\log\left(\frac{x^2}{n}\right)=-4\sum_{\gamma'}\frac{\sin^2(\gamma'\log x)}{(\gamma')^2}+J_2, \tag{30}$$

where $\frac{1}{2}+i\gamma'$ runs over the non-trivial zeros of $L(s,\pi_F\times\pi_G,\rho_4\otimes\rho_4)$ and

$$J_2=\frac{1}{2\pi i}\int_{(1/2)}\left(\frac{G_2'}{G_2}(s)+\frac{G_2'}{G_2}(1-s)\right)\left(\frac{x^{s-\frac{1}{2}}-x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\right)^2 ds.$$

Here, $G_2(s)$ is the archimedean part of $L(s,\pi_F\times\pi_G,\rho_4\otimes\rho_4)$. By [IK04, Proposition 5.7], we can show that

$$\sum_{\gamma}\frac{\sin^2(\gamma\log x)}{\gamma^2},\quad \sum_{\gamma'}\frac{\sin^2(\gamma'\log x)}{(\gamma')^2}\ll\log(k_1k_2)(\log x)^2.$$

By Stirling's formula, we can show that J_1 and J_2 are also bounded by

$$O(\log(k_1k_2)(\log x)^2).$$

Suppose that $\Lambda_{F\times F}(n)=\Lambda_{F\times G}(n)$ for all $n<x^2$. Subtracting (30) from (29) implies

$$0=8(x-2+x^{-1})+O((\log k_1k_2)(\log x)^2). \tag{31}$$

This will give a contradiction when $x\gg(\log k_1k_2)(\log\log k_1k_2)^2$. That is, if F is not a multiple of G, then we can find a sufficiently large C such that, for some integer $n\le C(\log k_1k_2)^2(\log\log k_1k_2)^4$, $\Lambda_{F\times F}(n)\ne\Lambda_{F\times G}(n)$. Then Theorem 1.4 can be deduced by the following lemma. □

Lemma 6.1. *Assume the notations above. Suppose that we can find A such that $\Lambda_{F\times F}(n)\ne\Lambda_{F\times G}(n)$ for some $n\le A$. Then we can find $n\le A$ such that $a_F(n)\ne a_G(n)$. Moreover, for such $n\le A$ we have $\tilde\lambda_F(n)\ne\tilde\lambda_G(n)$.*

Proof. As for the first assertion, notice that $\Lambda_{F\times F}(n)$ and $\Lambda_{F\times G}(n)$ are arithmetic functions supported on prime powers. So there exist a prime number p and a positive integer r such that $p^r\le A$, and $\Lambda_{F\times F}(p^r)\ne\Lambda_{F\times G}(p^r)$. We consider two cases: when $p\ge\lfloor A^{1/2}\rfloor+1$ and $p\le\lfloor A^{1/2}\rfloor$. Here, $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.

In the first case, $p^2>A$ and hence $\Lambda_{F\times F}(p)\ne\Lambda_{F\times G}(p)$. It can be shown that $\Lambda_{F\times F}(p)=a_F(p)^2\log p$ and $\Lambda_{F\times G}(p)=a_F(p)a_G(p)\log p$. Therefore, we have $a_F(p)\ne a_G(p)$.

In the second case, we prove by contradiction. Suppose that $a_F(n)=a_G(n)$ for $n\le A$. Then for $p\le\lfloor A^{1/2}\rfloor$, we have $a_F(p^i)=a_G(p^i)$ for $i=1,2$. This implies that F and G have the same Satake parameters at p (up to permutation), which can be obtained by (18) and (15)-(16). This shows that $\Lambda_{F\times F}(p^r)=\Lambda_{F\times G}(p^r)$ for any r, which is a contradiction.

The second assertion immediately follows from the relation between $a_F(n)$ and $\tilde\lambda_F(n)$; see (18). □

Remark 6.2. *Suppose that*

$$L(s,\pi_F,\rho_5)=\sum_{n=1}^{\infty}\frac{b_F(n)}{n^s}\quad\textit{and}\quad L(s,\pi_G,\rho_5)=\sum_{n=1}^{\infty}\frac{b_G(n)}{n^s}.$$

Assume that $L(s,\pi_F\times\pi_G,\rho_5\otimes\rho_5)$ and $L(s,\pi_F\times\pi_F,\rho_5\otimes\rho_5)$ satisfies the Generalized Riemann Hypothesis. A similar argument will show: if F is not a scalar multiplication of G, then there exists an integer

$$n\ll(\log k_1k_2)^2(\log\log k_1k_2)^4$$

such that $b_F(n)\ne b_G(n)$. Indeed, a direct calculation will show that, $\{b_F(p^r)\}_{r=1}^{\infty}$ will determined by $\{b_F(p),b_F(p^2)\}$ and hence we can obtain a result similar to the first assertion as in Lemma 6.1.

Appendix A: Archimedean factors associated to certain L-functions

In this section, we would briefly discuss the calculation of the archimedean factors associated to the L-functions in Sect. 6.

Let $F \in \mathcal{S}_{k_1}(\Gamma_2)$ and $G \in \mathcal{S}_{k_2}(\Gamma_2)$ be Hecke eigenforms. Then we can associate the automorphic cuspidal representations π_F (resp. π_G) for F (resp. G) of $\mathrm{GSp}(4, \mathbb{A})$. For π_F, we can associate the completed spinor L-function and the completed standard L-function, denoted by $\Lambda(s, \pi_F, \rho_4)$ and $\Lambda(s, \pi_F, \rho_5)$, respectively. Moreover, via the Langlands transfer (see [PSS14, § 5.1]), we can find Π_4^F (resp. Π_5^F), which is an automorphic cuspidal representation of $\mathrm{GL}(4, \mathbb{A})$ (resp. $\mathrm{GL}(5, \mathbb{A})$) such that

$$\Lambda(s, \pi_F, \rho_4) = \Lambda(s, \Pi_4^F) \quad \text{and} \quad \Lambda(s, \pi_F, \rho_5) = \Lambda(s, \Pi_5^F).$$

In this case, the Rankin-Selberg L-function $\Lambda(s, \pi_F \times \pi_G, \rho_4 \otimes \rho_4)$ and $\Lambda(s, \pi_F \times \pi_G, \rho_5 \otimes \rho_5)$ is defined by the Rankin-Selberg convolutions on $\mathrm{GL}(4) \times \mathrm{GL}(4)$ and $\mathrm{GL}(5) \times \mathrm{GL}(5)$, respectively, i.e.,

$$\Lambda(s, \pi_F \times \pi_G, \rho_4 \otimes \rho_4) = \Lambda(s, \Pi_4^F \times \Pi_4^G) \quad \text{and} \quad \Lambda(s, \pi_F \times \pi_G, \rho_5 \otimes \rho_5) = \Lambda(s, \Pi_5^F \times \Pi_5^G). \tag{32}$$

To calculate the associated archimedean factors, we recall some basic facts regarding the real Weil group $W_\mathbb{R} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times$. Here, the multiplication on $\mathbb{C}^\times$ is standard, and j is an element satisfying $j^2 = -1$ and $jzj^{-1} = \bar{z}$ (complex conjugation) for $z \in \mathbb{C}^\times$. More precisely, we are considering representations of $W_\mathbb{R}$, which are continuous homomorphisms $W_\mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ for some n with the image consisting of semisimple elements. By [Kna94], every finite-dimensional semisimple representation of $W_\mathbb{R}$ is completely recucible, and each irreducible representation is either one- or two-dimensional. The complete list of one-dimensional representations is as follows:

$$\varphi_{+,t} \colon re^{i\theta} \longmapsto r^{2t}, \quad j \mapsto 1, \tag{33}$$

$$\varphi_{-,t} \colon re^{i\theta} \longmapsto r^{2t}, \quad j \mapsto -1, \tag{34}$$

where $t \in \mathbb{C}$, and we write any non-zero complex number z as $re^{i\theta}$ with $r \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. The two-dimensional representations are precisely

$$\varphi_{\ell,t} \colon re^{i\theta} \mapsto \begin{bmatrix} r^{2t}e^{i\ell\theta} & \\ & r^{2t}e^{-i\ell\theta} \end{bmatrix}, \qquad j \mapsto \begin{bmatrix} & (-1)^\ell \\ 1 & \end{bmatrix}, \tag{35}$$

where $\ell \in \mathbb{Z}_{>0}$ and $t \in \mathbb{C}$. And the corresponding L-factors, i.e., the archimedean factors, are given as follows:

$$L_\infty(s, \varphi) = \begin{cases} \Gamma_\mathbb{R}(s+t) & \text{if } \varphi = \varphi_{+,t}, \\ \Gamma_\mathbb{R}(s+t+1) & \text{if } \varphi = \varphi_{-,t}, \\ \Gamma_\mathbb{C}(s+t+\frac{\ell}{2}) & \text{if } \varphi = \varphi_{\ell,t}. \end{cases} \tag{36}$$

Here,

$$\Gamma_\mathbb{R}(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right), \qquad \Gamma_\mathbb{C}(s) := 2(2\pi)^{-s}\Gamma(s), \tag{37}$$

where $\Gamma(s)$ is the usual gamma function. By a direct calculations we have the following lemma:

Lemma A.1. *For $\ell, \ell_1, \ell_2 \in \mathbb{Z}_{>0}$ and $t_1, t_2 \in \mathbb{C}$, we have*

$$\varphi_{+,t_1} \otimes \varphi_{+,t_2} = \varphi_{-,t_1} \otimes \varphi_{-,t_2} = \varphi_{+,t_1+t_2} \tag{38}$$

$$\varphi_{+,t_1}\otimes\varphi_{-,t_2}=\varphi_{-,t_1}\otimes\varphi_{+,t_2}=\varphi_{-,t_1+t_2} \tag{39}$$
$$\varphi_{\pm,t_1}\otimes\varphi_{\ell,t_2}=\varphi_{\ell,t_1+t_2} \tag{40}$$
$$\varphi_{\ell_1,t_1}\otimes\varphi_{\ell_2,t_2}=\begin{cases}\varphi_{\ell_1+\ell_2,t_1+t_2}\oplus\varphi_{|\ell_1-\ell_2|,t_1+t_2} & \text{if } \ell_1\neq\ell_2,\\ \varphi_{\ell_1+\ell_2,t_1+t_2}\oplus\varphi_{+,t_1+t_2}\oplus\varphi_{-,t_1+t_2} & \text{if } \ell_1=\ell_2.\end{cases} \tag{41}$$

Remark A.2. *Recall that $\Gamma_{\mathbb{C}}(s)=\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$, the second case in (41) looks precisely like the first case in (41), if we allow $\ell_1=\ell_2$.*

For our purpose, we will only consider the case $t=0$; in this case, we write $\varphi_\pm$ instead of $\varphi_{\pm,0}$ and φ_ℓ instead of $\varphi_{\ell,0}$. It follows from [Sch17, § 3.2] (observing that $\lambda_1=k_1-1$ and $\lambda_2=k_1-2$) and [PSS14, Theorem 5.1.2] that the L-parameter of Π_4^F at the archimedean place is given by:

$$\varphi_{2k_1-3}\oplus\varphi_1. \tag{42}$$

Proposition A.3. *Assume the notations above. The archimedean place of $L(s,\pi_F\times\pi_G,\rho_4\otimes\rho_4)$ is given by:*

$$\Gamma_{\mathbb{C}}(s+k_1+k_2-3)\,\Gamma_{\mathbb{C}}(s+k_1-1)\,\Gamma_{\mathbb{C}}(s+k_2-1)\Gamma_{\mathbb{C}}(s+k_1-2)\,\Gamma_{\mathbb{C}}(s+k_2-2)$$
$$\Gamma_{\mathbb{C}}(s+1)\Gamma_{\mathbb{C}}(s+|k_1-k_2|)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1).$$

Proof. Without of loss of generality, we can assume that $k_1\geq k_2$. The associating L-parameter at the archimedean place is

$$(\varphi_{2k_1-3}\oplus\varphi_1)\otimes(\varphi_{2k_2-3}\oplus\varphi_1)=(\varphi_{2k_1-3}\otimes\varphi_{2k_2-3})\oplus(\varphi_{2k_1-3}\otimes\varphi_1)\oplus(\varphi_{2k_2-3}\otimes\varphi_1)\oplus(\varphi_1\otimes\varphi_1). \tag{43}$$

Using Lemma A.1, then (43) becomes

$$\begin{cases}\varphi_{2(k_1+k_2)-6}\oplus\varphi_{2(k_1-k_2)}\oplus\varphi_{2k_1-2}\oplus\varphi_{2k_1-4}\oplus\varphi_{2k_2-2}\oplus\varphi_{2k_2-4}\oplus\varphi_2\oplus\varphi_+\oplus\varphi_- & \text{if } k_1>k_2,\\ \varphi_{4k-6}\oplus\varphi_+\oplus\varphi_-\oplus\varphi_{2k-2}\oplus\varphi_{2k-4}\oplus\varphi_{2k-2}\oplus\varphi_{2k-4}\oplus\varphi_2\oplus\varphi_+\oplus\varphi_- & \text{if } k=k_1=k_2.\end{cases}$$

Hence the desired result follows from the Remark A.2 above. ∎

On the other hand, by the construction of the standard L-function, we have that the L-parameter of Π_5^F is

$$\varphi_{2k_1-2}\oplus\varphi_{2k_1-4}\oplus\varphi_+. \tag{44}$$

Here, we note that $k_1\geq 2$; see [Sch17, Table 5]. A similar argument as in the proof of Proposition A.3 implies:

Proposition A.4. *Assume the notations above. The archimedean place of $\Lambda(s,\pi_F\times\pi_G,\rho_5\otimes\rho_5)$ is given by:*

$$\Gamma_{\mathbb{C}}(s+k_1+k_2-2)\,\Gamma_{\mathbb{C}}(s+|k_1-k_2|)^2\,\Gamma_{\mathbb{C}}(s+k_1+k_2-3)^2\,\Gamma_{\mathbb{C}}(s+|k_2-k_1-1|)\,\Gamma_{\mathbb{C}}(s+k_1-1)$$
$$\Gamma_{\mathbb{C}}(s+|k_1-k_2-1|)\,\Gamma_{\mathbb{C}}(s+k_1+k_2-4)\,\Gamma_{\mathbb{C}}(s+k_1-2)\,\Gamma_{\mathbb{C}}(s+k_2-1)\,\Gamma_{\mathbb{C}}(s+k_2-2)\,\Gamma_{\mathbb{R}}(s).$$

Again, by Remark A.2 we can write $\Gamma_{\mathbb{C}}(s+0)$ as $\Gamma_{\mathbb{R}}(s)\,\Gamma_{\mathbb{R}}(s+1)$ if happens.

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